Internet Appendix

Macroeconomic Factors in Oil Futures Markets

November 14, 2017

1 Model Specification

Consider a Gaussian model where the log spot price s_t of a commodity depends on N_L spanned state variables L_t, which may be latent or observed, and N_M unspanned state variables M_t that are observed:

$$
\begin{aligned}
\begin{bmatrix} L_{t+1} \\ M_{t+1} \end{bmatrix} &= K^{\mathbb{P}}_{0X} + K^{\mathbb{P}}_{1X} X_t + \Sigma_X \epsilon^{\mathbb{P}}_{t+1} \\
L_{t+1} &= K^{\mathbb{Q}}_{0L} + K^{\mathbb{Q}}_{1L} L_t + \Sigma_L \epsilon^{\mathbb{Q}}_{t+1} \\
s_t &= \delta_0 + \delta_1' L_t
\end{aligned}
\tag{1}
$$

where

- $\mathbb{P}$ denotes dynamics under the physical measure
- $\mathbb{Q}$ denotes dynamics under the risk neutral measure
- $\epsilon^{\mathbb{Q}}_{L,t+1} \sim N(0,\, I_{N_L})$, $\epsilon^{\mathbb{P}}_{t+1} \sim N(0,\, I_N)$
- Σ_L is the top left $N_L \times N_L$ block of Σ_X; Σ_L, Σ_X are lower triangular

(1) is equivalent to specifying the equation for s_t and the $\mathbb{P}$-dynamics plus a lognormal affine discount factor with 'essentially affine' prices of risk as in Duffee (2002). For $N_M = 0$ the framework includes models such as Gibson and Schwartz (1990); Schwartz (1997); Schwartz and Smith (2000); Casassus and Collin-Dufresne (2005) as special cases (see Appendix 1.4). Standard recursions show that (1) implies affine log prices for futures,

$$
f_t = A + BL_t \tag{2}
$$

$$
f_t = \begin{bmatrix} f_t^1 & f_t^2 & \dots & f_t^J \end{bmatrix}'
$$

where f_t^j is the price of a j period future and J is the number of futures maturities.

Estimating the model as written presents difficulties; with two spanned factors and two macro factors there are 40 free parameters, and different sets of parameter values may be observationally equivalent due to rotational indeterminacy. Discussing models of the form (1) for bond yields, Hamilton and Wu (2012) refer to *"tremendous numerical challenges in estimating the necessary parameters from the data due to highly nonlinear and badly behaved likelihood surfaces."* In general, affine futures pricing models achieve identification by specifying dynamics that are less general than (1).

Joslin, Singleton and Zhu (2011); Joslin, Priebsch and Singleton (2014) show that if N_L linear combinations of bond yields are measured without error then any term structure model of the form (1) is equivalent to a model with those N_L factors in place of the latent factors. They construct a minimal parametrization where no sets of parameters are redundant - models in the "JPS form" are unique. Thus the likelihood surface is well behaved and contains a single global maximum. Their results hold to a very close approximation if the linear combinations of yields are observed with relatively small and idiosyncratic errors.

Section 2 demonstrates the same result for futures markets: if N_L linear combinations of log futures prices are measured without error,

$$\mathcal{P}_t = W f_t \tag{3}$$

for any full rank $N_L \times J$ matrix W, then any model of the form (1) is observationally equivalent to a unique model of the form

$$
\begin{bmatrix} \Delta\mathcal{P}_{t+1} \\ \Delta UM_{t+1} \end{bmatrix} = \begin{array}{rcl} \Delta Z_{t+1} & = & K_0^{\mathbb{P}} + K_1^{\mathbb{P}} Z_t + \Sigma_Z \epsilon_{t+1}^{\mathbb{P}} \\ \Delta\mathcal{P}_{t+1} & = & K_0^{\mathbb{Q}} + K_1^{\mathbb{Q}} \mathcal{P}_t + \Sigma_{\mathcal{P}} \epsilon_{t+1}^{\mathbb{Q}} \\ s_t & = & \rho_0 + \rho_1 \mathcal{P}_t \end{array} \tag{4}
$$

parametrized by $\theta = (\lambda^{\mathbb{Q}},\ p_\infty,\ \Sigma_Z,\ K_0^{\mathbb{P}},\ K_1^{\mathbb{P}})$, where

- λ^Q are the N_L ordered eigenvalues of $K_1^{\mathbb{Q}}$
- p_∞ is a scalar intercept
- Σ_Z is the lower triangular Cholesky decomposition of the covariance matrix of innovations in the state variables
- $\Sigma_{\mathcal{P}}\Sigma_{\mathcal{P}}' = [\Sigma_Z\Sigma_Z']_{N_L}$, the top left $N_L \times N_L$ block of $\Sigma_Z\Sigma_Z'$

1.1 $\mathcal{P}_t$ Measured Without Error

In this paper I assume that while each of the log futures maturities is observed with iid measurement error, the pricing factors $\mathcal{P}_t^1$ and $\mathcal{P}_t^2$ are measured without error.

$$
f_t^j = A_j + B_j \mathcal{P}_t + \nu_t^j,\ \nu_t^j \sim N(0, \zeta_j^2)
$$

The use of the first two PCs of log price levels is not important: in unreported results I find that all estimates and results are effectively identical using other alternatives such as the first two PCs of log price changes or of returns, or a priori weights such as

$$W = \begin{bmatrix} 1 & 1 & 1 & ... & 1 \\ 1 & 2 & 3 & ... & 12 \end{bmatrix}$$

The identifying assumption that N_L linear combinations of yields are measured without error is commonly used in the literature. Given the model parameters, values of the latent factors at each date are then extracted by inverting the relation (2). In unreported results I find that all estimates and results are effectively identical if I allow the pricing factors to be measured with error and instead estimate them via the Kalman filter.

1.2 Rotating to s_t and c_t

Once the model is estimated in the JPS form, I rotate $(\mathcal{P}_t^1, \mathcal{P}_t^2)$ to be the model implied log spot price and instantaneous cost of carry, $(s_t,\, c_t)$. For s_t this is immediate:

$$s_t = \rho_0 + \rho_1 \mathcal{P}_t$$

For c_t the definition is as follows. Any agent with access to a storage technology can buy the spot commodity, sell a one month future, store for one month and make delivery. Add up all the costs and benefits of doing so (including interest, costs of storage, and convenience yield) and express them as quantity c_t where the total cost in dollar terms $= S_t(e^{c_t} - 1)$. Then in the absence of arbitrage it must be the case that

$$F_t^1 = S_t e^{c_t}$$

$$f_t^1 = s_t + c_t = E^{\mathbb{Q}}\left[s_{t+1}\right] + \frac{1}{2}\sigma_s^2$$

$$c_t = E^{\mathbb{Q}}\left[\Delta s_{t+1}\right] + \frac{1}{2}\sigma_s^2$$

$$= \rho_1[K_0^{\mathbb{Q}} + K_1^{\mathbb{Q}}\mathcal{P}_t] + \frac{1}{2}\sigma_s^2$$

1.3 Risk Premiums

Szymanowska et al. (2014) define the per-period log basis as

$$y_t^n \equiv f_t^n - s_t$$

They define the futures spot premium as

$$\pi_{s,t} \equiv E_t\left[s_{t+1} - s_t\right] - y_t^1$$

and the term premium as

$$\pi_{y,t}^n \equiv y_t^1 + (n-1)E_t\left[y_{t+1}^{n-1}\right] - ny_t^n$$

In our framework, the spot premium can be expressed as

$$\pi_{s,t} \equiv E_t\left[s_{t+1} - s_t\right] - y_t^1$$

$$= E_t^{\mathbb{P}}\left[s_{t+1}\right] - f_t^1 = E_t^{\mathbb{P}}\left[s_{t+1}\right] - E_t^{\mathbb{Q}}\left[s_{t+1}\right] - \frac{1}{2}\sigma_s^2$$

$$= \Lambda_t^s - \frac{1}{2}\sigma_s^2$$

In our framework, the term premium for $n = 2$ (the smallest n for which a term premium exists) can be expressed as

$$\pi_{y,t}^n \equiv y_t^1 + (n-1)E_t\left[y_{t+1}^{n-1}\right] - ny_t^n$$

$$\begin{aligned}\pi_{y,t}^2 &= f_t^1 - s_t + E_t^{\mathbb{P}}\left[f_{t+1}^1 - s_{t+1}\right] - 2 \times \frac{1}{2}(f_t^2 - s_t) \\ &= f_t^1 + E_t^{\mathbb{P}}[s_{t+1} + c_{t+1}] - E_t^{\mathbb{P}}[s_{t+1}] - E^{\mathbb{Q}}[s_{t+1} + c_{t+1}] - \frac{1}{2}\sigma_{f_{t+1}^1}^2 \\ &= E^{\mathbb{Q}}[s_{t+1}] + \frac{1}{2}\sigma_{s_{t+1}}^2 + E_t^{\mathbb{P}}[s_{t+1} + c_{t+1}] - E_t^{\mathbb{P}}[s_{t+1}] - E^{\mathbb{Q}}[s_{t+1} + c_{t+1}] - \frac{1}{2}\sigma_{f_{t+1}^1}^2 \\ &= \Lambda_t^c + \left(\frac{1}{2}\sigma_{s_{t+1}}^2 - \frac{1}{2}\sigma_{f_{t+1}^1}^2\right)\end{aligned}$$

Thus the spot premium and term premium of Szymanowska et al. (2014) correspond *exactly* to the risk premiums in our model Λ_t^s and Λ_t^c respectively, minus a Jensen term in each case which in our framework is constant.

1.4 Comparison with other Futures Pricing Models

The model (1) is a canonical form, so any affine Gaussian model is nested by it. For example, the Gibson and Schwartz (1990); Schwartz (1997); Schwartz and Smith (2000) two factor model in discrete time is the following:

$$\begin{bmatrix} \Delta s_{t+1} \\ \Delta \delta_{t+1} \end{bmatrix} = \begin{bmatrix} \mu \\ \kappa\alpha \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & -\kappa \end{bmatrix} \begin{bmatrix} s_t \\ \delta_t \end{bmatrix} + \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}^{1/2} \epsilon_{t+1}^{\mathbb{P}} \tag{5}$$

$$\begin{bmatrix} \Delta s_{t+1} \\ \Delta \delta_{t+1} \end{bmatrix} = \begin{bmatrix} r \\ \kappa\alpha - \lambda \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & -\kappa \end{bmatrix} \begin{bmatrix} s_t \\ \delta_t \end{bmatrix} + \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}^{1/2} \epsilon_{t+1}^{\mathbb{Q}} \tag{6}$$

which is clearly a special case of (1).

The Casassus and Collin-Dufresne (2005) model in discrete time is:

$$\begin{bmatrix} \Delta X_{t+1} \\ \Delta\hat{\delta}_{t+1} \\ \Delta r_{t+1} \end{bmatrix} = \begin{bmatrix} \kappa_X^P \theta_X^P + \kappa_{Xr}^P \theta_r^P + \kappa_{X\hat{\delta}}^P \theta_{\hat{\delta}}^P \\ \kappa_{\hat{\delta}}^P \theta_{\hat{\delta}}^P \\ \kappa_r^P \theta_r^P \end{bmatrix} + \begin{bmatrix} -\kappa_X^P & -\kappa_{X\hat{\delta}}^P & -\kappa_{Xr}^P \\ 0 & -\kappa_{\hat{\delta}}^P & 0 \\ 0 & 0 & -\kappa_r^P \end{bmatrix} \begin{bmatrix} X_t \\ \hat{\delta}_t \\ r_t \end{bmatrix} + \begin{bmatrix} \sigma_X & 0 & 0 \\ 0 & \sigma_{\hat{\delta}} & 0 \\ 0 & 0 & \sigma_r \end{bmatrix} \begin{bmatrix} 1 & & \\ \rho_{X\delta} & 1 & \\ \rho_{Xr} & \rho_{\delta r} & 1 \end{bmatrix}^{1/2} \epsilon_{t+1}^{\mathbb{P}} \tag{7}$$

$$\begin{bmatrix} \Delta X_{t+1} \\ \Delta\hat{\delta}_{t+1} \\ \Delta r_{t+1} \end{bmatrix} = \begin{bmatrix} \alpha_X \theta_X^Q + (\alpha_r - 1)\theta_r^Q + \theta_{\hat{\delta}}^Q \\ \kappa_{\hat{\delta}}^Q \theta_{\hat{\delta}}^Q \\ \kappa_r^Q \theta_r^Q \end{bmatrix} + \begin{bmatrix} -\alpha_X & -1 & 1-\alpha_r \\ 0 & -\kappa_{\hat{\delta}}^Q & 0 \\ 0 & 0 & -\kappa_r^Q \end{bmatrix} \begin{bmatrix} X_t \\ \hat{\delta}_t \\ r_t \end{bmatrix} + \begin{bmatrix} \sigma_X & 0 & 0 \\ 0 & \sigma_{\hat{\delta}} & 0 \\ 0 & 0 & \sigma_r \end{bmatrix} \begin{bmatrix} 1 & & \\ \rho_{X\delta} & 1 & \\ \rho_{Xr} & \rho_{\delta r} & 1 \end{bmatrix}^{1/2} \epsilon_{t+1}^{\mathbb{Q}} \tag{8}$$

(see their formulas 7, 12, 13 and 27, 28, 30).

2 JPS Parametrization

I assume that N_L linear combinations of log futures prices are measured without error,

$$\mathcal{P}_t^L = W f_t$$

for any full-rank real valued $N_L \times J$ matrix W, and show that any model of the form

$$\begin{aligned}
\begin{bmatrix} \Delta L_{t+1} \\ \Delta M_{t+1} \end{bmatrix} = \Delta X_{t+1} &= K_{0X}^{\mathbb{P}} + K_{1X}^{\mathbb{P}} X_t + \Sigma_X \epsilon_{t+1}^{\mathbb{P}} \\
\Delta L_{t+1} &= K_{0L}^{\mathbb{Q}} + K_{1L}^{\mathbb{Q}} X_t + \Sigma_L \epsilon_{L,t+1}^{\mathbb{Q}} \\
s_t &= \delta_0 + \delta_1' X_t
\end{aligned} \tag{9}$$

is observationally equivalent to a unique model of the form

$$\begin{aligned}
\begin{bmatrix} \Delta \mathcal{P}_{t+1}^L \\ \Delta M_{t+1} \end{bmatrix} = \Delta Z_{t+1} &= K_0^{\mathbb{P}} + K_1^{\mathbb{P}} Z_t + \Sigma_Z \epsilon_{Z,t+1}^{\mathbb{P}} \\
\Delta \mathcal{P}_{t+1}^L &= K_0^{\mathbb{Q}} + K_1^{\mathbb{Q}} Z_t + \Sigma_{\mathcal{P}} \epsilon_{t+1}^{\mathbb{Q}} \\
s_t &= \rho_0 + \rho_1' Z_t
\end{aligned} \tag{10}$$

which is parametrized by $\theta = (\lambda^{\mathbb{Q}},\ p_\infty,\ \Sigma_Z,\ K_0^{\mathbb{P}},\ K_1^{\mathbb{P}})$.

The proof follows that of Joslin, Priebsch and Singleton (2014). Joslin, Singleton and Zhu (2011) solves with no macro factors over all cases including zero, repeated and complex eigenvalues.

Assume the model (9) under consideration is nonredundant, that is, there is no observationally equivalent model with fewer than N state variables. If there is such a model, switch to it and proceed.

2.1 Observational Equivalence

Given any model of the form (9), the $J \times 1$ vector of log futures prices f_t is affine in L_t,

$$f_t = A_L + B_L L_t$$

Hence the set of N_L linear combinations of futures prices, $\mathcal{P}_t^L$, is as well:

$$\mathcal{P}_t^L = W_L f_t = W_L A_L + W_L B_L L_t$$

Assume that the N_L ordered elements of $\lambda^{\mathbb{Q}}$, the eigenvalues of $K_{1L}^{\mathbb{Q}}$, are real, distinct and nonzero. There exists a matrix C such that $K_{1L}^{\mathbb{Q}} = C\text{diag}(\lambda^{\mathbb{Q}})C^{-1}$. Define $D = Cdiag(\delta_1)C^{-1}$, $D^{-1} = Cdiag(\delta_1)^{-1}C^{-1}$ and

$$Y_t = D[L_t + \left(K_{1L}^{\mathbb{Q}}\right)^{-1} K_{0L}^{\mathbb{Q}}]$$

$$\Rightarrow \; L_t = D^{-1}Y_t - \left(K_{1L}^{\mathbb{Q}}\right)^{-1} K_{0L}^{\mathbb{Q}}$$

Then

$$\Delta Y_{t+1} = D\Delta L_{t+1}$$

$$= D[K_{0L}^{\mathbb{Q}} + K_{1L}^{\mathbb{Q}}(D^{-1}Y_t - \left(K_{1L}^{\mathbb{Q}}\right)^{-1} K_{0L}^{\mathbb{Q}}) + \Sigma_L \epsilon_{L,t+1}^{\mathbb{Q}}]$$

$$= \text{diag}(\lambda^{\mathbb{Q}})Y_t + D\Sigma_L \epsilon_{L,t+1}^{\mathbb{Q}}$$

and

$$\begin{bmatrix} \Delta Y_{t+1} \\ \Delta M_{t+1} \end{bmatrix} = \begin{bmatrix} D & 0 \\ 0 & I_{\mathcal{M}} \end{bmatrix} [K^{\mathbb{P}}_{0X} + K^{\mathbb{P}}_{1X}(\begin{bmatrix} D^{-1} & 0 \\ 0 & I_{\mathcal{M}} \end{bmatrix} \begin{bmatrix} Y_t \\ M_t \end{bmatrix} - \begin{bmatrix} \left(K^{\mathbb{Q}}_{1L}\right)^{-1} K^{\mathbb{Q}}_{0L} \\ 0 \end{bmatrix}) + \Sigma_X \epsilon^{\mathbb{P}}_{t+1}]$$

$$= K^{\mathbb{P}}_{0Y} + K^{\mathbb{P}}_{1Y} \begin{bmatrix} Y_t \\ M_t \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & I_{\mathcal{M}} \end{bmatrix} \Sigma_X \epsilon^{\mathbb{P}}_{t+1}$$

and

$$p_t = \delta_0 + \delta_1' L_t = \delta_0 + \delta_1' D^{-1} Y_t - \delta_1' \left(K^{\mathbb{Q}}_{1L}\right)^{-1} K^{\mathbb{Q}}_{0L} = p_\infty + \iota \cdot Y_t$$

where ι is a row of N_L ones.

$$f_t = A_Y + B_Y Y_t$$

$$\mathcal{P}^L_t = W f_t = W A_Y + W B_Y Y_t$$

The model is nonredundant $\Rightarrow$ WB_Y is invertible:

$$Y_t = (WB_Y)^{-1} \mathcal{P}^L_t - (WB_Y)^{-1} W A_Y$$

$$\cdot\mathcal{P}^L_{t+1} = WB_Y \Delta Y_{t+1} = WB_Y \text{diag}(\lambda^{\mathbb{Q}})[(WB_Y)^{-1}\mathcal{P}^L_t - (WB_Y)^{-1} W A_Y] + WB_Y D \Sigma_L \epsilon^{\mathbb{Q}}_{L,t+1}$$

$$= K^{\mathbb{Q}}_0 + K^{\mathbb{Q}}_1 \mathcal{P}^L_t + \Sigma_{\mathcal{P}} \epsilon^{\mathbb{Q}}_{t+1}$$

Further,

$$\Delta Z_{t+1} = \begin{bmatrix} \cdot\mathcal{P}^L_{t+1} \\ \Delta M_{t+1} \end{bmatrix} = \begin{bmatrix} WB_Y & 0 \\ 0 & I_{\mathcal{M}} \end{bmatrix} \begin{bmatrix} \Delta Y_{t+1} \\ \Delta M_{t+1} \end{bmatrix}$$

$$= \begin{bmatrix} WB_Y & 0 \\ 0 & I_{\mathcal{M}} \end{bmatrix} \left(K_{0Y}^{\mathbb{P}} + K_{1Y}^{\mathbb{P}} \begin{bmatrix} Y_t \\ M_t \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & I_{\mathcal{M}} \end{bmatrix} \Sigma_X \epsilon_{t+1}^{\mathbb{P}} \right)$$

$$= K_0^{\mathbb{P}} + K_1^{\mathbb{P}} Z_t + \Sigma_Z \epsilon_{t+1}^{\mathbb{P}}$$

$$p_t = p_\infty + \iota \cdot Y_t = p_\infty + \iota \cdot (WB_Y)^{-1} \mathcal{P}_t^L - \iota \cdot (WB_Y)^{-1} W A_Y = \rho_0 + \rho_1' \mathcal{P}_t^L$$

Collecting the formulas: given any model of the form (1), there is an observationally equivalent model of the form (4), parametrized by $\theta = (\lambda^{\mathbb{Q}},\ p_\infty,\ \Sigma_Z,\ K_0^{\mathbb{P}},\ K_1^{\mathbb{P}})$, where

- $D = Cdiag(\delta_1)^{-1} C^{-1}$
- $\Sigma_Z = \begin{bmatrix} WB_Y D & 0 \\ 0 & I_{\mathcal{M}} \end{bmatrix} \Sigma_X$, $\Sigma_{\mathcal{P}} = [\Sigma_Z]_{\mathcal{L}\mathcal{L}}$
- $B_Y = \begin{bmatrix} \iota'[I_{\mathcal{L}+\mathcal{M}} + \text{diag}(\lambda^{\mathbb{Q}})] \\ \vdots \\ \iota'[I_{\mathcal{L}+\mathcal{M}} + \text{diag}(\lambda^{\mathbb{Q}})]^J \end{bmatrix}$
- $A_Y = \begin{bmatrix} p_\infty + \frac{1}{2}\iota' \Sigma_{\mathcal{P}} \Sigma_{\mathcal{P}}' \iota \\ \vdots \\ A_{Y,J-1} + \frac{1}{2} B_{Y,J-1} \Sigma_{\mathcal{P}} \Sigma_{\mathcal{P}}' B_{Y,J-1}' \end{bmatrix}$
- $K_1^{\mathbb{Q}} = WB_Y \text{diag}(\lambda^{\mathbb{Q}})(WB_Y)^{-1}$, $K_0^{\mathbb{Q}} = -K_1^{\mathbb{Q}} W A_Y$
- $\rho_0 = p_\infty - \iota \cdot (WB_Y)^{-1} W A_Y$, $\rho_1' = \iota \cdot (WB_Y)^{-1}$

In estimation I adopt the alternate form

- $\Delta Y_{t+1} = \begin{bmatrix} p_\infty \\ 0 \end{bmatrix} + \text{diag}(\lambda^{\mathbb{Q}}) Y_t + D \Sigma_X \epsilon_{t+1}^{\mathbb{Q}}$

- $p_t = \iota \cdot Y_t$
- $A_Y = \begin{bmatrix} p_\infty + \frac{1}{2}\iota' \Sigma_{\mathcal{P}} \Sigma'_{\mathcal{P}} \iota \\ \vdots \\ A_{Y,J-1} + B_{Y,J-1} \begin{bmatrix} p_\infty \\ 0 \end{bmatrix} + \frac{1}{2} B_{Y,J-1} \Sigma_{\mathcal{P}} \Sigma'_{\mathcal{P}} B'_{Y,J-1} \end{bmatrix}$
- $K_1^{\mathbb{Q}} = W B_Y \text{diag}(\lambda^{\mathbb{Q}})(W B_Y)^{-1}$, $K_0^{\mathbb{Q}} = W B_Y \begin{bmatrix} p_\infty \\ 0 \end{bmatrix} - K_1^{\mathbb{Q}} W A_Y$
- $\rho_0 = -\iota \cdot (W B_Y)^{-1} W A_Y$, $\rho_1' = \iota \cdot (W B_Y)^{-1}$

which is numerically stable when $\lambda^{\mathbb{Q}}(1) \to 0$. See the online supplement to JSZ 2011.

2.2 Uniqueness

We consider two models of the form (4) with parameters θ and $\hat{\theta} = (\hat{\lambda}^{\mathbb{Q}},\ \hat{p}_\infty,\ \hat{\Sigma}_Z,\ \hat{K}_0^{\mathbb{P}},\ \hat{K}_1^{\mathbb{P}})$ that are observationally equivalent and show that this implies $\theta = \hat{\theta}$.

Since $Z_t = \begin{bmatrix} \mathcal{P}_t^L \\ M_t \end{bmatrix}$ are all observed, $\{\Sigma_Z,\ K_0^{\mathbb{P}},\ K_1^{\mathbb{P}}\} = \{\hat{\Sigma}_Z,\ \hat{K}_0^{\mathbb{P}},\ \hat{K}_1^{\mathbb{P}}\}$.

Since $f_t = A + B Z_t$ are observed, $A(\theta) = A(\hat{\theta})$, $B(\theta) = B(\hat{\theta})$.

Suppose $\lambda^{\mathbb{Q}} \neq \hat{\lambda}^{\mathbb{Q}}$. Then by the uniqueness of the ordered eigenvalue decomposition,

$$B_Y^j(\lambda) \neq B_Y^j(\hat{\lambda})\, \forall j$$

$$\Rightarrow W B_Y(\lambda) \neq W B_Y(\hat{\lambda}) \ \Rightarrow (W B_Y(\lambda))^{-1} \neq (W B_Y(\hat{\lambda}))^{-1}$$

$$\Rightarrow \rho_1(\lambda) \neq \rho_1(\hat{\lambda}) \ \Rightarrow B(\lambda) \neq B(\hat{\lambda})$$

, a contradiction. Hence $\lambda^{\mathbb{Q}} = \hat{\lambda}^{\mathbb{Q}}$. Then $A(\lambda^{\mathbb{Q}}, p^\infty) = A(\hat{\lambda}^{\mathbb{Q}}, \hat{p}^\infty) \ \Rightarrow p^\infty = \hat{p}^\infty$.

3 Estimation

Given the futures prices and macroeconomic time series $\{f_t,\ M_t\}_{t=1,\ldots,T}$ and the set of portfolio weights W that define the pricing factors:

$$\mathcal{P}_t = W f_t$$

we need to estimate the minimal parameters $\theta = (\lambda^{\mathbb{Q}},\ p_\infty,\ \Sigma_Z,\ K_0^{\mathbb{P}},\ K_1^{\mathbb{P}})$ in the JPS form. The estimation is carried out by maximum likelihood (MLE). If no restrictions are imposed (i.e. we are estimating the canonical model (9)), then $K_0^{\mathbb{P}}$, $K_1^{\mathbb{P}}$ do not affect futures pricing and are estimated consistently via OLS. Otherwise $K_0^{\mathbb{P}}$, $K_1^{\mathbb{P}}$ are obtained by GLS taking the restrictions into account. The OLS estimate of Σ_Z is used as a starting value, and the starting value for p_∞ is the unconditional average of the nearest-maturity log futures price. Both were always close to their MLE value. Finally we search over a range of values for the eigenvalues $\lambda^{\mathbb{Q}}$.

After the MLE estimate of the model in the JPS form is found, we rotate and translate the spanned factors from $\mathcal{P}_t^1$, $\mathcal{P}_t^2$ to s_t, c_t as described in 1.2. we rotate and translate UM_t to M_t, so that the estimate reflects the behavior of the time series M_t:

$$\begin{bmatrix} s_t \\ c_t \\ M_t \end{bmatrix} = \begin{bmatrix} \rho_0 \\ \frac{1}{2}\sigma_s^2 + \rho_1 K_0^{\mathbb{Q}} \\ \alpha_{M\mathcal{P}} \end{bmatrix} + \begin{bmatrix} \rho_1 & 0_{1\times N_M} \\ \rho_1 K_1^{\mathbb{Q}} & 0_{1\times N_M} \\ 0_{N_M\times 1} & \beta_{M\mathcal{P}} \end{bmatrix} \begin{bmatrix} \mathcal{P}_t \\ UM_t \end{bmatrix}$$

where

$$M_t = \alpha_{M\mathcal{P}} + \beta_{M\mathcal{P}}\mathcal{P}_t + UM_t$$

4 Robustness Checks

4.1 Alternative Measures of Real Activity

The predictability I find using the Chicago Fed National Activity Index also holds using other forward-looking measures of real activity. In this section I show that the same results obtain using the Aruoba-Diebold-Scotti (ADS)[1] index or the Conference Board's Leading Economic Index (LEI)[2] in place of the CFNAI.

The LEI is a weighted forward-looking index of real activity like the CFNAI, but uses different weights and macroeconomic time series. The ADS index is a real-time forward-looking index of real activity that is extracted by filtering from a third set of macroeconomic time series. The time series are similar because all three are intended as forward-looking measures of real activity, but they are not identical: the correlation between the ADS index and the CFNAI is 83.8% in levels and 58.7% in changes while the correlation between the LEI and the CFNAI is 8.6% in levels and 25.3% in changes.

Table 1 shows the results of the return forecasting regressions using the ADS index, and Table 1 using the LEI. We see that both alternative indices forecast oil futures returns and prices in the same directions as the CFNAI, conditional on the information in the oil futures curve.

Table 3 shows the feedback matrix $K_1^{\mathbb{P}}$ implied by estimating the affine model using the ADS index or the LEI in place of the CFNAI. Both the ADS index and the LEI forecast a higher spot price of oil (top right) and the spot price of oil negatively forecasts a lower value of both indices (bottom left). Thus, the main conclusions are the same using alternative measures of real activity.

[1] https://www.philadelphiafed.org/research-and-data/real-time-center/business-conditions-index/
[2] https://www.conference-board.org/data/bcicountry.cfm?cid=1

Table 1: Panel A shows the results of forecasting the returns to the short-roll and 3 month excess-holding strategies in oil futures. Panel B shows the results of forecasting changes in the principal components of log futures prices. The forecasting variables are 1) three sets of 'reduced-form' state variables $\mathcal{P}_t$ based on oil futures prices and 2) the Aruoba-Diebold-Scotti index ADS_t plus log oil inventory INV_t. The data are monthly from from 1/1986 to 6/2014. Newey-West standard errors with 6 lags are in parentheses.

Panel A: Forecasting Returns

$$r_{t+1} = \alpha + \beta_{ADS,INV} M_t + \beta_{\mathcal{P}} \mathcal{P}_t + \epsilon_{t+1}$$

	Short Roll Return			Excess Holding Return		
ADS_t	0.0314**	0.0291**	0.0290*	−0.0023**	−0.0018*	−0.0017*
	(0.0141)	(0.0144)	(0.0148)	(0.0011)	(0.0010)	(0.0009)
INV_t	0.0197	0.0215	0.0166	−0.0030	−0.0062	−0.0068
	(0.0915)	(0.0917)	(0.0890)	(0.0105)	(0.0096)	(0.0096)
Spanned Factors $\mathcal{P}_t$:	$PC^{1,2}$	PC^{1-5}	f^{1-12}	$PC^{1,2}$	PC^{1-5}	f^{1-12}
T	341	341	341	339	339	339
Adj. $R^2(\mathcal{P}_t)$	0.4%	0.7%	4.6%	5.5%	9.4%	10.3%
Adj. $R^2(\mathcal{P}_t + M_t)$	3.7%	3.3%	7.1%	7.6%	10.8%	11.6%

Panel B: Forecasting PCs

$$\Delta PC_{t+1} = \alpha + \beta_{ADS,INV} M_t + \beta_{\mathcal{P}} \mathcal{P}_t + \epsilon_{t+1}$$

	ΔPC^1			ΔPC^2		
ADS_t	0.084*	0.081*	0.082*	0.0108**	0.0098**	0.0100**
	(0.043)	(0.044)	(0.046)	(0.0049)	(0.0047)	(0.0046)
INV_t	0.0031	-0.0161	-0.0346	0.0339	0.0418	0.0392
	(0.2499)	(0.2462)	(0.2422)	(0.0549)	(0.0495)	(0.0463)
Spanned Factors $\mathcal{P}_t$:	$PC^{1,2}$	PC^{1-5}	f^{1-12}	$PC^{1,2}$	PC^{1-5}	f^{1-12}
T	341	341	341	341	341	341
Adjusted $R^2(\mathcal{P}_t)$	-0.4%	-0.5%	2.9%	6.5%	8.0%	10.3%
Adj. $R^2(\mathcal{P}_t + M_t)$	2.6%	2.1%	5.6%	7.6%	9.0%	11.3%

Table 2: Panel A shows the results of forecasting the returns to the short-roll and 3 month excess-holding strategies in oil futures. Panel B shows the results of forecasting changes in the principal components of log futures prices. The forecasting variables are 1) three sets of 'reduced-form' state variables $\mathcal{P}_t$ based on oil futures prices and 2) the Leading Economic Index (LEI_t) plus log oil inventory INV_t. The data are monthly from from 1/1986 to 6/2014. Newey-West standard errors with 6 lags are in parentheses.

Panel A: Forecasting Returns

$$r_{t+1} = \alpha + \beta_{LEI,INV} M_t + \beta_{\mathcal{P}} \mathcal{P}_t + \epsilon_{t+1}$$

	Short Roll Return			Excess Holding Return		
LEI_t	0.172*	0.182**	0.174**	−0.0049	−0.0050	−0.0040
	(0.090)	(0.084)	(0.083)	(0.0081)	(0.0062)	(0.0059)
INV_t	0.179	0.182	0.173	−0.0089	−0.0118	−0.0118
	(0.134)	(0.131)	(0.128)	(0.0131)	(0.0113)	(0.0108)
Spanned Factors $\mathcal{P}_t$:	$PC^{1,2}$	PC^{1-5}	f^{1-12}	$PC^{1,2}$	PC^{1-5}	f^{1-12}
T	341	341	341	339	339	339
Adj. $R^2(\mathcal{P}_t)$	0.4%	0.7%	4.6%	5.5%	9.4%	10.3%
Adj. $R^2(\mathcal{P}_t + M_t)$	2.3%	2.7%	6.3%	5.4%	9.5%	10.4%

Panel B: Forecasting PCs

$$\Delta PC_{t+1} = \alpha + \beta_{LEI,INV} M_t + \beta_{\mathcal{P}} \mathcal{P}_t + \epsilon_{t+1}$$

	ΔPC^1			ΔPC^2		
LEI_t	0.513**	0.544**	0.525**	0.0842**	0.0584	0.0597
	(0.253)	(0.243)	(0.239)	(0.0369)	(0.0365)	(0.0368)
INV_t	0.467	0.457	0.428	0.1074	0.0938	0.0928
	(0.380)	(0.376)	(0.373)	(0.0580)	(0.0533)	(0.0502)
Spanned Factors $\mathcal{P}_t$:	$PC^{1,2}$	PC^{1-5}	f^{1-12}	$PC^{1,2}$	PC^{1-5}	f^{1-12}
T	341	341	341	341	341	341
Adjusted $R^2(\mathcal{P}_t)$	-0.4%	-0.5%	2.9%	6.5%	8.0%	10.3%
Adj. $R^2(\mathcal{P}_t + M_t)$	1.9%	1.8%	5.0%	8.0%	8.7%	11.0%

Table 3: Maximum likelihood (ML) estimates of the macro-finance model for Nymex crude oil futures, using data from 1/1986 to 6/2014. s, c are the spot price and annualized cost of carry respectively. ADS and LEI are the Aruoba-Diebold-Scotti index and the Conference Board Leading Economic Index respectively. INV is the log of the private U.S. crude oil inventory as reported by the EIA. The coefficients are over a monthly horizon, and the state variables are de-meaned. ML standard errors are in parentheses.

Panel A: Aruoba-Diebold-Scotti (ADS) Index

	$K_1^{\mathbb{P}}$		
	s_t	c_t	ADS_t
Δs_{t+1}	-0.004	0.059**	0.031***
	(0.008)	(0.027)	(0.009)
Δc_{t+1}	0.014*	−0.127***	−0.019**
	(0.008)	(0.025)	(0.009)
ΔADS_{t+1}	−0.069**	0.079	−0.264***
	(0.033)	(0.107)	(0.036)

Panel B: Conference Board Leading Economic Index (LEI)

	$K_1^{\mathbb{P}}$		
	s_t	c_t	LEI_t
Δs_{t+1}	−0.028**	0.061**	0.126**
	(0.011)	(0.027)	(0.054)
Δc_{t+1}	0.028***	−0.128***	-0.074
	(0.010)	(0.025)	(0.050)
ΔLEI_{t+1}	−0.002**	-0.001	0.003
	(0.001)	(0.002)	(0.003)

4.2 Excluding the Financial Crisis

Inspecting the data, we question whether the results in the paper are driven by a few influential observations – in particular the huge swings in oil prices and real activity during 2008-2009. Table 4 presents the forecasting regressions estimated on a subsample from January 1986 to December 2007. We see that the conclusions are the same, and indeed the forecasting power of *GRO* is slightly *stronger* when we omit 2008-2014.

Table 5 presents the full model estimated on the subsample from January 1986 to December 2007. The subsample estimate is similar to the full-sample estimate, and the key coefficients of ΔGRO_{t+1} on s_t and Δs_{t+1} on GRO_t remain statistically significant.

4.3 Time Varying Volatility

This section examines the results of the forecasting regressions when we add measures of time-varying volatility in oil futures. If volatility drives a higher hedge premium, then volatility might be an omitted factor that explains the positive association between real activity and the oil price forecast. I examine three standard volatility measures: $optvol_t$ is the implied volatility from short-term options on oil futures, $garchvol_t$ is the conditional volatility of Δf^1_{t+1} estimated as a GARCH(1,1) process, and $sqchg_t$ is the lagged squared change $(\Delta f^1_t)^2$ of the nearby log futures price.

Table 6 shows that the crude oil volatility indexes are indeed negatively correlated with *GRO*. However, time-varying volatility does not forecast oil prices or returns, and thus does not explain the forecasting power of real activity. Table 7 shows that none of the volatility factors is significant in the forecasting regressions, none of them significantly raises the adjusted R^2, and (most importantly) their inclusion does not alter the forecasting power of real activity.

Table 4: Panel A shows the results of forecasting the returns to the short-roll and 3 month excess-holding strategies in oil futures. Panel B shows the results of forecasting changes in the principal components of log futures prices. The forecasting variables are 1) three sets of 'reduced-form' state variables $\mathcal{P}_t$ based on oil futures prices and 2) the real activity index GRO_t and log oil inventory INV_t. The data are monthly from from 1/1986 to 12/2007. Newey-West standard errors with six lags are in parentheses.

Panel A: Forecasting Futures Returns

$$r_{t+1} = \alpha + \beta_{GRO,INV} M_t + \beta_{\mathcal{P}} \mathcal{P}_t + \epsilon_{t+1}$$

	Short Roll Return			Excess Holding Return		
GRO_t	0.0300***	0.0281***	0.0249**	−0.0015*	−0.0013	−0.0013
	(0.0089)	(0.0092)	(0.0096)	(0.0009)	(0.0009)	(0.0009)
INV_t	-0.026	-0.022	-0.018	0.0167	0.0124	0.0116
	(0.120)	(0.115)	(0.104)	(0.123)	(0.120)	(0.122)
Spanned Factors $\mathcal{P}_t$:	$PC^{1,2}$	PC^{1-5}	f^{1-12}	$PC^{1,2}$	PC^{1-5}	f^{1-12}
T	263	263	263	263	263	263
Adj. $R^2(\mathcal{P}_t)$	-0.5%	-0.2%	5.6%	10.6%	12.6%	12.7%
Adj. $R^2(\mathcal{P}_t + M_t)$	2.3%	2.0%	7.1%	12.8%	13.6%	13.5%

Panel B: Forecasting PCs

$$\Delta PC_{t+1} = \alpha + \beta_{GRO,INV} M_t + \beta_{\mathcal{P}} \mathcal{P}_t + \epsilon_{t+1}$$

	ΔPC^1			ΔPC^2		
GRO_t	0.0845***	0.0810***	0.0729***	0.0067	0.0067	0.0056
	(0.0227)	(0.0236)	(0.0243)	(0.0057)	(0.0057)	(0.0060)
INV_t	-0.014	-0.051	-0.039	-0.002	0.004	0.001
	(0.296)	(0.270)	(0.248)	(0.092)	(0.084)	(0.078)
Spanned Factors $\mathcal{P}_t$:	$PC^{1,2}$	PC^{1-5}	f^{1-12}	$PC^{1,2}$	PC^{1-5}	f^{1-12}
T	263	263	263	263	263	263
Adjusted $R^2(\mathcal{P}_t)$	-0.5%	-0.3%	5.6%	7.4%	8.5%	10.8%
Adj. $R^2(\mathcal{P}_t + M_t)$	2.7%	2.6%	7.7%	7.1%	8.3%	10.4%

Table 5: Maximum likelihood (ML) estimate of the macro-finance model for Nymex crude oil futures using data from 1/1986 to 12/2007. s, c are the spot price and annualized cost of carry respectively. GRO is the monthly Chicago Fed National Activity Index. INV is the log of the private U.S. crude oil inventory as reported by the EIA. The coefficients are over a monthly horizon, and the state variables are de-meaned. ML standard errors are in parentheses.

	$K_0^{\mathbb{P}}$	$K_1^{\mathbb{P}}$			
		s_t	c_t	GRO_t	INV_t
s_{t+1}	0.011	-0.005	0.058	0.029***	-0.005
	(0.007)	(0.014)	(0.036)	(0.011)	(0.106)
c_{t+1}	-0.012	0.026*	−0.130***	-0.012	0.003
	(0.007)	(0.014)	(0.036)	(0.011)	(0.108)
GRO_{t+1}	0.057	−0.181**	0.651***	−0.580***	-1.653
	(0.035)	(0.070)	(0.178)	(0.054)	(0.526)
INV_{t+1}	0.003	−0.007*	0.033***	-0.003	−0.125***
	(0.002)	(0.004)	(0.009)	(0.003)	(0.027)

	$K_0^{\mathbb{Q}}$	$K_1^{\mathbb{Q}}$	
		s_t	c_t
s_{t+1}	-0.003	0.000	0.083***
	(0.007)	(0.005)	(0.011)
c_{t+1}	0.000	-0.009	−0.113***
	(0.012)	(0.014)	(0.031)

Shock Volatilities
[off-diagonal = % correlations]

	s	c	GRO	INV
s	0.102			
c	-84%	0.056		
GRO	7%	-1%	0.499	
INV	-20%	29%	2%	0.024

Table 6: The table shows the correlations of the monthly real activity index GRO and three indexes of time varying volatility in crude oil prices. The time series are monthly from 1/1989 to 6/2014 and have been demeaned. $garchvol_t$ is the conditional volatility of Δf_{t+1}^1 estimated as a GARCH(1,1) process. $optvol_t$ is the implied volatility based on the prices of at-the-money options on one month futures. $sqchg_t$ is the squared change $(\Delta f_t^1)^2$ of the front-month futures contract last month.

	GRO_t	$sqchg_t$	$optvol_t$	$garchvol_t$
GRO_t	1			
$sqchg_t$	-24.8%	1		
$optvol_t$	-54.9%	50.7%	1	
$garchvol_t$	-51.8%	27.6%	68.7%	1

4.4 Year-on-Year Changes

Although futures returns are a stationary process, they may contain slow-moving components i.e. time varying expected returns or regime shifts that are effectively nonstationary over a monthly horizon. Log futures prices f_t and the principal components portfolios $\mathcal{P}t$ that summarize them are themselves nonstationary or very close to it. In this setting, forecasting regressions may have poor small-sample properties.

To address this concern I rerun the forecasting regressions after transforming f_t and PC_t into year-on-year changes. The macro variables M_t are not transformed as they are strongly stationary in the first place, and year-on-year differencing would eliminate the important variation in GRO (i.e. at business cycle frequency). Table 8 shows that after removing persistence in the regressors, the incremental forecasting power of real activity for futures returns and changes in the level factor is effectively unchanged.

Table 7: The table shows the results of forecasting returns to oil futures including measures of time-varying volatility. The data are monthly from 1/1986 to 6/2014 except *optvol* which is monthly from 1/1989 to 6/2014. The forecasting variables are GRO_t, and the first two PCs of log oil futures prices, and three measures of crude oil volatility. $optvol_t$ is the implied volatility based on the prices of at-the-money options on one month futures. $garchvol_t$ is the conditional volatility of Δf^1_{t+1} estimated as a GARCH(1,1) process. $sqchg_t$ is the lagged squared change $(\Delta f^1_t)^2$ of the log price of the first nearby futures contract. Newey-West standard errors with six lags are in parentheses.

Panel A: Forecasting Futures Returns

$$r_{t+1} = \alpha + \beta_{GRO} M_t + \beta_{\mathcal{P}} PC^{1,2}_t + \beta_{VOL} VOL_t + \epsilon_{t+1}$$

	Short Roll Return			Excess Holding Return		
GRO_t	0.023**	0.027***	0.029**	−0.0010	−0.0022**	−0.0018**
	(0.010)	(0.009)	(0.012)	(0.0009)	(0.0008)	(0.0008)
$optvol_t$	-0.009			0.0047*		
	(0.018)			(0.0024)		
$garchvol_t$		-0.167			0.0574	
		(0.432)			(0.0474)	
$sqchg_t$			0.002			0.0016
			(0.008)			(0.0014)
T	295	341	341	293	339	339
Adj. $R^2(\mathcal{P}_t + GRO_t)$	4.1%	4.5%	4.5%	7.4%	9.7%	9.7%
Adj. $R^2(\mathcal{P}_t + GRO_t + VOL_t)$	3.9%	4.3%	4.2%	11.2%	11.1%	11.5%

Panel B: Forecasting PCs

$$\Delta PC_{t+1} = \alpha + \beta_{GRO} M_t + \beta_{\mathcal{P}} PC^{1,2}_t + \beta_{VOL} VOL_t + \epsilon_{t+1}$$

	ΔPC^1			ΔPC^2		
GRO_t	0.063**	0.067***	0.076	−0.0093*	0.0107***	0.0113**
	(0.028)	(0.025)	(0.034)	(0.0054)	(0.0035)	(0.0046)
$optvol_t$	-0.011			-0.0011		
	(0.048)			(0.0090)		
$garchvol_t$		-0.852			0.094	
		(1.071)			(0.273)	
$sqchg_t$			0.0095			0.0024
			(0.0197)			(0.0034)
T	295	341	341	295	341	341
Adj. $R^2(\mathcal{P}_t + GRO_t)$	2.8%	3.0%	3.0%	7.2%	8.5%	8.5%
Adj. $R^2(\mathcal{P}_t + GRO_t + VOL_t)$	2.5%	3.1%	2.8%	6.8%	8.3%	8.3%

Table 8: Panel A shows the results of forecasting the returns to the short-roll and 3 month excess-holding strategies in oil futures. Panel B shows the results of forecasting changes in the principal components of log futures prices. The forecasting variables are 1) three sets of year-on-year changes in the spanned state variables based on oil futures prices and 2) the real activity index GRO_t and log oil inventory INV_t. The data are monthly from from 1/1986 to 6/2014. Newey-West standard errors with six lags are in parentheses.

Panel A: Forecasting Futures Returns

$$r_{t+1} = \alpha + \beta_{GRO,INV} M_t + \beta_{\mathcal{P}} \left(\mathcal{P}_t^{1,2} - \mathcal{P}_{t-12}^{1,2} \right) + \epsilon_{t+1}$$

	Short Roll Return			Excess Holding Return		
GRO_t	0.0283***	0.0277**	0.0271**	−0.0034***	−0.0032***	−0.0029***
	(0.0109)	(0.0112)	(0.0114)	(0.0009)	(0.0009)	(0.0009)
INV_t	-0.039	-0.048	-0.050	0.0029	0.0022	0.0024
	(0.079)	(0.072)	(0.070)	(0.0070)	(0.0065)	(0.0064)
Spanned Factors $\mathcal{P}_t$:	$PC^{1,2}$	PC^{1-5}	f^{1-12}	$PC^{1,2}$	PC^{1-5}	f^{1-12}
T	329	329	329	327	327	327
Adj. $R^2(\mathcal{P}_t)$	2.4%	2.2%	1.2%	3.2%	4.9%	7.0%
Adj. $R^2(\mathcal{P}_t + M_t)$	6.5%	5.9%	4.6%	10.5%	10.9%	11.7%

Panel B: Forecasting PCs

$$\Delta PC_{t+1} = \alpha + \beta_{GRO,INV} M_t + \beta_{\mathcal{P}} \left(\mathcal{P}_t^{1,2} - \mathcal{P}_{t-12}^{1,2} \right) + \epsilon_{t+1}$$

	ΔPC^1			ΔPC^2		
GRO_t	0.095**	0.091**	0.088*	0.010	0.009	0.008
	(0.043)	(0.045)	(0.046)	(0.006)	(0.007)	(0.007)
INV_t	-0.231	-0.270	-0.287	0.051	0.034	0.030
	(0.281)	(0.254)	(0.251)	(0.060)	(0.056)	(0.055)
Spanned Factors $\mathcal{P}_t$:	$PC^{1,2}$	PC^{1-5}	f^{1-12}	$PC^{1,2}$	PC^{1-5}	f^{1-12}
T	329	329	329	329	329	329
Adjusted $R^2(\mathcal{P}_t)$	5.7%	5.7%	7.7%	9.0%	9.5%	11.5%
Adj. $R^2(\mathcal{P}_t + M_t)$	8.4%	8.1%	10.0%	9.4%	9.6%	11.5%

Table 9: Parameters of the calibration for computing real option values

	$K_0^{\mathbb{P}}$	$K_1^{\mathbb{P}}$		
		s_t	c_t	GRO_t
s_{t+1}	0.00	1.00	0.083	0.03
c_{t+1}	0.00	0.00	0.90	0.00
GRO_{t+1}	0.00	-0.10	0.00	0.60

Σ	s	c	GRO
s	0.10		
c	-0.08	0.06	
GRO	0	0	0.50

	$K_0^{\mathbb{Q}}$	$K_1^{\mathbb{Q}}$		
		s_t	c_t	GRO_t
s_{t+1}	0.00	1.00	0.083	0.00
c_{t+1}	0.00	0.00	0.90	0.00
GRO_{t+1}	$-\lambda$	-0.10	0.00	0.60

5 Real Option Valuation – Details

I model the log lifting cost (per-barrel cost of extraction) as

$$l_t = \kappa_l + 0.1 s_t + 0.01 GRO_t + \epsilon_t^l,\ \ \epsilon_t^l \sim N(0, \sigma_l)$$

That is, l_t varies with both s_t and GRO_t as well as having an i.i.d. idiosyncratic component with volatility σ_l. The other parameters in the simulated data are in Table 9. Notice the third row of $K_1^{\mathbb{Q}}$, which was not present in the model estimates. Pricing assets with payoffs that depend on M_t requires the risk neutral dynamics of M_t. In principle one could estimate the risk neutral dynamics of M_t with a tracking portfolio for GRO, but for simplicity I assume that exposure to GRO carries a fixed risk premium of λ.

I compute option values for different starting values of the lifting cost $L_0 = exp(l_0)$, with $S_0 = exp(s_0)$ equal to $80 per barrel and $c_0 = 0$. This simulates an oil firm evaluating wells that differ in their current lifting cost, conditional on a spot price of $80 and a flat futures curve.

References

Casassus, J. and Collin-Dufresne, P., Stochastic Convenience Yield implied from Commodity Futures and Interest Rates. Journal of Finance, 60 2005:5, pp. 2283–2331 ⟨URL: http://onlinelibrary.wiley.com/doi/10.1111/j.1540-6261.2005.00799.x/abstract⟩.

Duffee, G. R., Term Premia and Interest Rate Forecasts in Affine Models. Journal of Finance, 57 2002:1, pp. 405–443 ⟨URL: http://onlinelibrary.wiley.com/doi/10.1111/1540-6261.00426/abstract⟩.

Gibson, R. and Schwartz, E. S., Stochastic Convenience Yield and the Pricing of Oil Contingent Claims. Journal of Finance, 45 1990, pp. 959–976 ⟨URL: http://www.jstor.org/stable/10.2307/2328801⟩.

Hamilton, James D. and Wu, Jing Cynthia, Identification and estimation of Gaussian affine term structure models. Journal of Econometrics, 168 2012:2, pp. 315–331 ⟨URL: https://doi.org/10.1016/j.jeconom.2012.01.035⟩.

Joslin, S., Priebsch, M. and Singleton, K.J., Risk Premiums in Dynamic Term Structure Models with Unspanned Macro Risks. Journal of Finance, 69 2014:3, pp. 1197–1233 ⟨URL: http://onlinelibrary.wiley.com/doi/10.1111/jofi.12131/abstract⟩.

Joslin, Scott, Singleton, Kenneth J. and Zhu, Haoxiang, A New Perspective on Gaussian Dynamic Term Structure Models. Review of Financial Studies, 24 2011:3, pp. 926–970 ⟨URL: https://doi.org/10.1093/rfs/hhq128⟩.

Schwartz, E. and Smith, J.E., Short-Term Variations and Long-Term Dynamics in

Commodity Prices. Management Science, 46 (7) 2000, pp. 893–911 ⟨URL: https://doi.org/10.1287/mnsc.46.7.893.12034⟩.

Schwartz, E.S., The Stochastic Behavior of Commodity Prices: Implications for Valuation and Hedging. Journal of Finance, 52 1997:3, pp. 923–973 ⟨URL: http://onlinelibrary.wiley.com/doi/10.1111/j.1540-6261.1997.tb02721.x/abstract⟩.

Szymanowska, Marta et al., An Anatomy of Commodity Futures Risk Premia. Journal of Finance, 69 2014:1, pp. 453–482 ⟨URL: http://dx.doi.org/10.1111/jofi.12096⟩.